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THE INDEPENDENCE RATIO AND GENUS OF A GRAPH

BY

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ABSTRACT. In this paper we study the relationship between the genus of a graph and the ratio of the independence number to the number of vertices.

I. Introduction. Suppose G is a simple graph with V vertices and E edges that can be embedded with F faces on S_n, a surface of genus n. A set of vertices I is independent in G if no pair of vertices in I is adjacent, and the largest number of vertices in any independent set we denote by $\alpha(G)$. Since our interest lies in the percentage of vertices in an independent set we define the *independence ratio*, $\mu(G) = \alpha(G)/V$. It is convenient to specify the sets

$$U(n) = \{\mu(G)\colon G \text{ embeds on } S_n\} \quad \text{and} \quad L(n) = \{\text{limit points of } U(n)\}.$$

Since any graph that embeds on S_j also embeds on S_k $(k > j)$, the following containments hold:

$$U(0) \subset U(1) \subset \dots \subset U(n) \subset \dots \quad \text{and}$$
$$L(0) \subset L(1) \subset \dots \subset L(n) \subset \dots .$$

The independence ratio of a graph is related to the chromatic number of a graph in a natural manner. If the vertices of G can be colored in r colors, the most popular color must be assigned to at least V/r of the vertices of G. For $n > 0$ denote the Heawood number by $H = H(n) = [\frac{7}{2} + \frac{1}{2}(48n + 1)^{1/2}]$. Since every graph that embeds on S_n can be colored in $H(n)$ colors we have

$$U(n) \subset [1/H, 1] \qquad (n > 0).$$

Since K_H embeds on S_n the above is the smallest interval that contains $U(n)$. Furthermore we have that $U(n)$ contains every rational between a fourth and one by the following.

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THEOREM 1.1. *Suppose p/q is rational with $1/4 \leq p/q \leq 1$. There exists a planar graph G with $V = q$ and $\alpha(G) = p$.*

PROOF. Induction on V.

COROLLARY. $L(0) \supset [1/4, 1]$.

REMARK. Erdös has conjectured that $U(0) \subset [\frac{1}{4}, 1]$ [6, p. 280]. This is ostensibly weaker than the Four Color Conjecture. One of the present authors has shown that $U(0) \subset (\frac{2}{9}, 1]$ [1], [2].

It is not difficult to show that $U(0)$ is dense in a subinterval of [0, 1]; that is, $\mathrm{cl}(U(0)) = L(0)$. One of the consequences of this paper is that zero is the only value of n for which this is true. However, it is easily shown that each $L(n)$ is an interval.

THEOREM 1.2. *There is a nonincreasing sequence of numbers $\langle a_n \rangle$ such that $L(n) = [a_n, 1]$ for all n.*

With this much as background we can state the goal of this paper: to investigate the isolated points that are the small elements of $U(n)$ and to get lower bounds for the a_n. Our efforts are directed toward the following.

CONJECTURE. $L(n) = L(0)$ *for all n.*

§II presents preliminaries from topological graph theory as well as induction steps commonly used in the remainder of the paper. In §III we exhibit the toroidal graphs with independence ratio less than a fifth and show that there are only essentially four such graphs. §IV discusses those graphs embedded on a given surface with small independence ratios. Finally §V presents a proof that $L(n)$ is contained in $[\frac{1}{5}, 1]$.

II. Some preliminaries. This section recalls some elementary facts concerning graphs embedded on S_n and presents inductive steps which will be used to prove several theorems.

The starting point for much of our work is Euler's Formula for connected graphs:

$$V - E + F \geq 2 - 2n.$$

From this we obtain the standard upper bound for the average degree:

$$2E/V \leq 6 + 12(n-1)/V.$$

Two immediate consequences of this bound are:

LEMMA 2.1. *A graph embedded on the torus either has a vertex of degree less than six or is regular of degree six.*

LEMMA 2.2. *If G, embedded on S_n $(n \geq 2)$, has at least $2H - 2$ vertices, then G has a vertex of degree less than $H - 1$.*

In the following we assume that $n \geq 1$ is fixed, G embeds on S_n and $H = H(n)$ is the Heawood number of S_n.

LEMMA 2.3. *If $G \neq K_H, K_{H-1}$ and $\mu(G) \leq 1/(H-1)$ (respectively $<$), then $V \geq 2H - 2$ (respectively $>$). Furthermore if $n \geq 2$, then G contains a vertex of degree less than $H - 1$.*

PROOF. If G is not a complete graph, $\alpha(G) \geq 2$. Thus $\mu(G) \leq 1/(H-1)$ implies that $V > H$. Since $K_{H+i} (i > 0)$ does not embed on S_n, for $H < V < 2H - 2$, $\mu(G) \geq 2/V > 1/(H-1)$. The proof for the strict inequalities is the same, and by the previous lemma in either case G contains a vertex of degree less than $H - 1$ when $n \geq 2$.

The presence of a vertex of small degree will often enable us to argue inductively. The next lemma presents a general induction step. For v a vertex of G, let $N(v)$ denote the neighbors of v, namely all vertices in G that are adjacent to v. Let $N^*(v)$ be $v + N(v)$ and $G - N^*(v)$ be the graph obtained from G by removing all of the vertices in $N^*(v)$. By $G(v)$ we shall mean the induced subgraph on the vertices of $N^*(v)$.

LEMMA 2.4. *Let r be a positive integer and s a real number $0 < s < 1$. Suppose G has a vertex x of degree less than r. Set $G' = G - N^*(x)$. If $rs \leq 1$ and $\mu(G') \geq s$ (respectively $>$), then $\mu(G) \geq s$ (respectively $>$). Further if q is a real number with $s < q < 1$ and $V \leq (1 - rs)/(q - s)$, then $\mu(G) \geq q$ (respectively $>$).*

PROOF. If the degree of x is less than r, then G' has at least $V - r$ vertices. By assumption $\alpha(G') \geq s(V - r)$. If I' is an independent set of vertices in G', then $I' + x$ is an independent set of vertices in G. Thus

$$\mu(G) \geq (1 + s(V - r))/V.$$

If $rs \leq 1$, $\mu(G) \geq s$. If we wish to show $\mu(G) \geq q$ we note that

$$(1 + s(V - r))/V \geq q$$

whenever $V \leq (1 - rs)/(q - s)$.

Typically we shall use this lemma when we have a vertex x of degree less than r in a graph whose independence ratio we wish to show to be at least $1/r$. We can often achieve this same independence ratio when we have a vertex x of degree r as follows.

Suppose x is a vertex in a graph G such that $N(x)$ contains two nonadjacent vertices a and b. We define $G(x, a, b)$ to be the graph obtained from G by deleting x and all of its neighbors except a and then adding edges from a to all of the remaining vertices which were previously adjacent to b. Clearly $G(x, a, b)$ can be embedded on S_n.

LEMMA 2.5. *Let r be a positive integer and s a real number, $0 < s < 1$.*

Suppose G has a vertex x of degree r and $G(x)$ is not K_{r+1}. If x has a pair of nonadjacent neighbors a and b with $\mu(G(x, a, b)) \geq s$ (respectively $>$) and $rs \leq 1$, then $\mu(G) \geq s$ (respectively $>$). Further if q is a real number with $s < q < 1$ and $V \leq (1 - rs)/(q - s)$, then $\mu(G) \geq q$ (respectively $>$).

Proof. Set $G' = G(x, a, b)$. By assumption $\alpha(G') \geq s(V - r)$. Suppose I' is an independent set of vertices in G'. If a is in I', then $I' + b$ is independent in G. If a is not in I', then $I' + x$ is independent in G. In either case we have $\mu(G) \geq (1 + s(V - r))/V$. If $rs \leq 1$, $\mu(G) \geq s$. As in the previous lemma if $V \leq (1 - rs)/(q - s)$, $\mu(G) \geq q$.

We need one more type of induction step. We shall say that a graph G, embedded on S_n, contains an $i \times j$ *configuration* if G has a triangular region bounded by (v_1, v_2, v_3) with v_1 of degree i and an adjacent triangular region bounded by (v_2, v_3, v_4) with v_4 of degree j. If G contains such a configuration we construct a new graph $G(v_2, v_3)$ as follows. We remove from G the vertices v_1, v_4, $N(v_1)$, and $N(v_4)$ except for v_2 and v_3. We then add edges joining v_2 to all vertices in G that were adjacent to any vertex in $N(v_1)$ and joining v_3 to all vertices in G that were adjacent to any vertex in $N(v_4)$. Figure 1 shows G and $G(v_2, v_3)$ in the case $i = j = 6$.

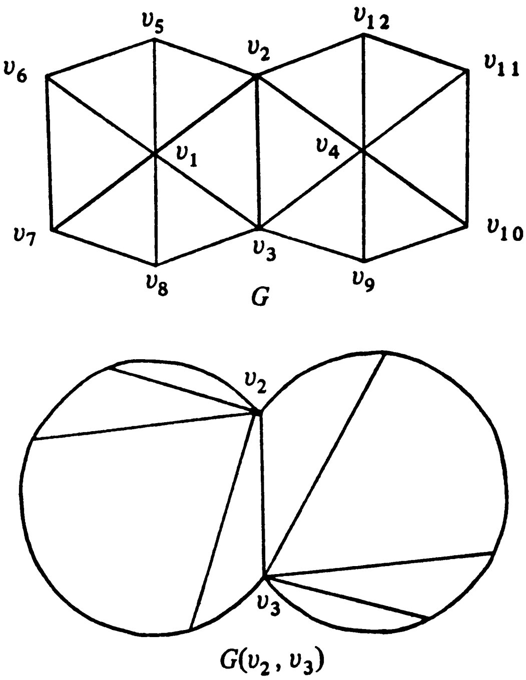

Figure 1

Lemma 2.6. *Suppose G contains a 6×6 configuration labelled as in Figure 1.*

Assume that every 3*-cycle in* G *is a face boundary. Let* $G' = G(v_2, v_3)$, s *a real number* $0 < s < 1$, *and* r *the number of distinct vertices in* $N(v_1) + N(v_4)$. *If* $\mu(G') \geq s$ (*respectively* $>$) *and* $rs \leq 2$, *then* $\mu(G) \geq s$ (*respectively* $>$). *Further if* $s < q < 1$ *and* $V \leq (2 - rs)/(q - s)$, *then* $\mu(G) \geq q$ (*respectively* $>$).

PROOF. G' has $V - r$ vertices since we have removed v_1, v_4, and all but two vertices of $N(v_1) + N(v_4)$. By assumption $\alpha(G') \geq s(V - r)$. Suppose I' is an independent set of vertices in G'. If neither v_2 nor v_3 is in I', then $I' + v_1 + v_4$ is independent in G. If v_2 is in I', we claim that $I' - v_2 + v_4 + v_5 + v_8$ is independent in G. Clearly v_4, v_5, and v_8 are adjacent to no other vertex of I'. If any pair of v_4, v_5, v_8 were adjacent, G would contain a 3-cycle that is not a face boundary. As above, if v_3 is in I' then $I' - v_3 + v_1 + v_9 + v_{12}$ is independent in G. In all cases we have $\mu(G) \geq (2 + s(V - r))/V$. If $rs \leq 2$, then $\mu(G) \geq s$. As before if $V \leq (2 - rs)/(q - s)$, then $\mu(G) \geq q$.

We continue §II with some more topological observations. Given an embedding of G on S_n, we say that a cycle C of G is contractible (respectively noncontractible or "n.c.") if C is (respectively is not) homotopic to a point. We then define $w(G)$, *the width of* G, as the length of the smallest n.c. cycle in G. Notice that if $w(G) > 3$, then every 3-cycle of G is contractible in the given embedding. We mean by the interior of such a 3-cycle C the subset of S_n which is bounded by C and is homeomorphic to a subset of the sphere. It is also convenient to define $hg(G)$, *the handle girth* of a graph G, as the minimum of $w(G)$ over all embeddings of G on S_n.

LEMMA 2.7. *If* G *is embedded on* S_n, $n \geq 1$, *and* G *contains a copy of* K_5, *then* $w(G) = 3$.

REMARK. By "G contains a copy of K_5" we mean that G contains five vertices, each pair of which is joined by an edge.

PROOF. Since K_5 is not planar the vertices of the K_5 in some order form a n.c. cycle. Suppose $(v_1, v_2, v_3, v_4, v_5)$ is n.c. Since v_1 is adjacent to v_3 either (v_1, v_2, v_3) is n.c. or (v_1, v_3, v_4, v_5) is n.c. Since v_1 is also adjacent to v_4 it must be that at least one of (v_1, v_2, v_3), (v_1, v_3, v_4) and (v_1, v_4, v_5) is n.c.

COROLLARY 2.8. *If* G, *embedded on* S_n, *contains a copy of* K_l $(l > 5)$, *then* G *less any* $l - 5$ *vertices is a graph also embedded with width* 3.

LEMMA 2.9. *Suppose* C *is a* 3*-cycle in* G *which is contractible; then* C *is a face boundary or* C *contains in its interior a vertex of degree less than six.*

PROOF. If C is contractible and not a face boundary then the subgraph of G induced by the vertices interior to and on C is planar. Suppose this subgraph contains r vertices, e edges, and j edges joining vertices of C with vertices interior to C. Then

$$2e \geq 6(r - 3) + 6 + j = 6r - 12 + j.$$

Euler's Formula for planar graphs yields $2e \leq 6r - 12$; thus $j = 0$. If $j = 0$, the vertices of G interior to C form a planar graph with average degree at least six which contradicts Euler's Formula for planar graphs.

Finally we investigate the size of the largest complete graph that can be embedded on S_n together with K_H. Suppose G embeds on S_n and contains K_H. We can think of the embedding of G as an embedding of K_H on S_n with the additional vertices of G contained in the regions determined by the embedding of K_H. We can determine bounds on the size of the genus of these regions.

Suppose that $H = H(n) = H(n + 1) = \ldots = H(n + j)$.

Thus

$$H \leq 7/2 + (48n + 1)^{1/2}/2 < H + 1 \tag{1}$$

and

$$H \leq 7/2 + (48(n + j) + 1)^{1/2}/2 < H + 1. \tag{2}$$

A region of S_n determined by an embedding of K_H can be homeomorphic to a subset of S_j provided that the above restrictions are met.

LEMMA 2.10. *Vertex disjoint copies of K_H and K_{H-i} do not embed on S_n for $i = 1, 2$.*

PROOF. Suppose j is the largest integer so that $H(n) = H(n + j)$. If $j = 0$ the lemma follows since the regions of K_H must be homeomorphic to subsets of the sphere. Thus the lemma is valid for $n \leq 6$. Suppose $j \geq 1$ and K_{H-i} embeds on S_j. Then

$$H - i \leq 7/2 + (48j + 1)^{1/2}/2 < H - i + 1. \tag{3}$$

For $i = 1$ equations (1) and (3) imply that

$$8H^2 - 64H + 129 \leq 48(n + j) + 1. \tag{4}$$

Combining (2) and (4) yields

$$8H^2 - 64H + 129 < 4H^2 - 20H + 25. \tag{5}$$

If $n \geq 2$ the inequality in (5) is not valid. Thus K_{H-1} cannot embed on S_j. If $i = 2$ a similar argument works for $n \geq 3$.

III. **The torus.** We separate the case $n = 1$ from the rest of our work because we have much more precise information about $U(1)$ than about $U(n)$ for $n > 1$. In particular we show in this section that with four exceptions $\mu(G) \geq \frac{1}{5}$ for toroidal graphs. The methods used are in most ways typical of those applied to the general case; however, the extra strength comes from Lemma 2.1. For the rest of this section we assume that G can be embedded on the torus.

LEMMA 3.1. *Suppose* $V \leq 10$. *If* $G \neq K_6, K_7$ *then* $\mu(G) \geq \frac{1}{5}$.

PROOF. If G is not a complete graph, $\alpha(G) \geq 2$; thus $\mu(G) \geq \frac{2}{10}$. If G is complete and neither K_6 nor K_7, $\mu(G) \geq \frac{1}{5}$ since K_8 does not embed on the torus.

LEMMA 3.2. *If* $V \geq 12$ *and* $hg(G) = 3$, *then* $\mu(G) \geq \frac{1}{5}$.

PROOF. Suppose G is embedded on the torus with C a n.c. 3-cycle. Construct G' from G by deleting the three vertices on C. G' is a planar graph which implies $\mu(G') > \frac{2}{9}$ [1]. Thus $\alpha(G) \geq \alpha(G') > \frac{2}{9}(V - 3)$.

If $V \geq 30$, $\frac{2}{9}\,(V - 3) \geq \frac{1}{5}V$; thus $\mu(G) \geq \frac{1}{5}$. If $V < 30$ it is well known that G' can be four colored [9]. Thus $\alpha(G) \geq \alpha(G') \geq \frac{1}{4}(V - 3)$.

If $V \geq 15$, $\frac{1}{4}(V - 3) \geq \frac{1}{5}V$; thus $\mu(G) \geq \frac{1}{5}$. If $V = 12, 13, 14$ then G' has at least nine vertices; hence $\alpha(G') \geq 3$. Thus $\mu(G) \geq \frac{3}{14} > \frac{1}{5}$.

Lemmas 3.1 and 3.2 omit the case of $V = 11$. We now consider this case in some detail and show that there are two classes of graphs with $V = 11$ and $\alpha(G) = 2$. By $K_7 + K_4$ we mean any graph with $V = 11$ which contains vertex disjoint copies of K_7 and K_4 with some edges joining the two cliques. Obviously $\mu(K_7 + K_4) = \frac{2}{11}$. The graph J shown in Figure 2 also has $\mu(J) = \frac{2}{11}$. Notice that J does not contain K_5; however, it contains lots of n.c. 3-cycles.

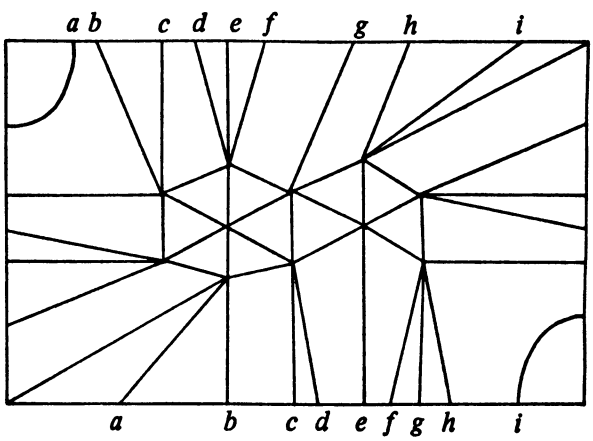

FIGURE 2

The next two lemmas show that if G is neither $K_7 + K_4$ nor J, then $\mu(G) \geq \frac{1}{5}$.

LEMMA 3.3. *If G is not regular of degree six*, $V = 11$, *and* $\mu(G) = \frac{2}{11}$, *then G is one of the graphs* $K_7 + K_4$.

PROOF. We show that if G is not a $K_7 + K_4$, then $\alpha(G) \geq 3$. Since G is not regular of degree six, G contains a vertex x of degree less than six. Since G is not a $K_7 + K_4$, one of $G(x)$ and $G' = G - N^*(x)$ is not complete. If G' is not complete, then Lemma 2.4 forces $\alpha(G) > 3$. If $G(x)$ is not complete, then Lemma 2.5 forces $\alpha(G) > 3$.

LEMMA 3.4. *If G is regular of degree six, $V = 11$, and $\mu(G) = \frac{2}{11}$, then G is isomorphic to J.*

PROOF. We sketch. Suppose v_1 and v_4 are nonadjacent vertices in G. If $N^*(v_1) + N^*(v_4)$ does not contain all vertices of G, then G contains three independent vertices. Hence $\#(N^*(v_1) \cap N^*(v_4)) = 3$. If these three vertices are independent we are done; if some pair is joined by an edge, G contains a 6×6 configuration (see Figure 1) with exactly one vertex from $v_5, \ldots, v_8$ identified with exactly one vertex from $v_9, \ldots, v_{12}$. By exhaustion one can show that any graph with these properties is isomorphic to J.

We can now proceed to the theorem.

THEOREM 3.5. *If G embeds on the torus and $G \neq K_7$, K_6, $K_7 + K_4$, and J, then $\mu(G) > \frac{1}{5}$.*

PROOF. The proof will be by induction on the number of vertices. We assume that if G' has fewer than V vertices, then either $\mu(G') > \frac{1}{5}$ or $G' = K_7$, K_6, $K_7 + K_4$ or J. By Lemmas 3.1, 3.3, and 3.4 we may assume $V > 12$.

Case 1. Suppose G contains a vertex x of degree less than five. Set $G' = G - N^*(x)$. If $\mu(G') < \frac{1}{5}$ then $G' = K_7$, K_6, $K_7 + K_4$ or J. In each of these cases G' contains a n.c. 3-cycle which implies that G contains a n.c. 3-cycle. By Lemma 3.2 $\mu(G) > \frac{1}{5}$. If $\mu(G') > \frac{1}{5}$, Lemma 2.4 implies $\mu(G) > \frac{1}{5}$.

Case 2. Suppose G contains a vertex x of degree five. If $G(x) = K_6$, Lemmas 2.7 and 3.2 force $\mu(G) > \frac{1}{5}$. If $G(x) \neq K_6$ then x has two nonadjacent neighbors, a and b. Set $G' = G(x, a, b)$. If $\mu(G') < \frac{1}{5}$ then $G' = K_7$, $K_6, K_7 + K_4$ or J. By Corollary 2.8 and the nature of J, $G' - a$ contains a n.c. 3-cycle; thus G contains a n.c. 3-cycle. Lemma 3.2 implies $\mu(G) > \frac{1}{5}$. If $\mu(G') > \frac{1}{5}$ Lemma 2.5 implies $\mu(G) > \frac{1}{5}$.

Case 3. Assume that G is regular of degree six. Such a graph on the torus is a triangulation. By Lemma 2.9 and because of Lemma 3.2, we may assume that every 3-cycle in G is a face boundary. Suppose (v_1, v_2, v_3) and (v_2, v_3, v_4) are two adjacent triangles and set $G' = G(v_2, v_3)$. If $G' = K_7$ or $K_7 + K_4$, $G' - v_2 - v_3$ contains K_5 and thus a n.c. 3-cycle. Hence G contains a n.c. 3-cycle. If $G' = J$, since J less any pair of adjacent vertices contains a n.c. 3-cycle so does G. If $G' = K_6$, $G' - v_2 - v_3$ is K_4. There are only two ways to embed K_4 on the torus: one gives rise to a 3-cycle that is not a face boundary, the other has a n.c. 3-cycle. Thus if $\mu(G') < \frac{1}{5}$, G contains a n.c. 3-cycle, and Lemma 3.2 forces $\mu(G) \geqslant \frac{1}{5}$. If $\mu(G') \geqslant \frac{1}{5}$, Lemma 2.6 implies $\mu(G) \geqslant \frac{1}{5}$.

COROLLARY 3.6. $U(1) \subset [\frac{1}{5}, 1] \cup \{\frac{2}{11}\} \cup \{\frac{1}{6}\} \cup \{\frac{1}{7}\}$.

COROLLARY 3.7. $L(1) \subset [\frac{1}{5}, 1]$.

IV. $U(n)$; $n \geqslant 2$. This section concerns itself with graphs with small independence ratios. In general we show that if the independence ratio is small enough the obstruction to finding a large independent set is a complete graph.

THEOREM 4.1. *If $G \neq K_H$ embeds on S_n $(n \geqslant 1)$, then $\mu(G) \geqslant 1/(H - 1)$.*

PROOF. For $n = 1$ this is a corollary of Theorem 3.5. Suppose G is a counterexample to the theorem with the fewest number of vertices, i.e. $\mu(G) < 1/(H - 1)$ and $G \neq K_H$. By Lemma 2.3, G must contain a vertex of degree less than $H - 1$.

Case 1. Suppose G contains a vertex x of degree less than $H - 2$. Set $G' = G - N^*(x)$. If $\mu(G') < 1/(H - 1)$, $G' = K_H$ and G had at most $2H - 2$ vertices contradicting Lemma 2.3. If $\mu(G') \geqslant 1/(H - 1)$, Lemma 2.4 implies $\mu(G) \geqslant 1/(H - 1)$.

Case 2. Suppose G contains a vertex x of degree $H - 2$. If $G(x) \neq K_{H-1}$ we may assume that two neighbors of x, say a and b, are not adjacent in G. Set $G' = G(x, a, b)$. If $\mu(G') < 1/(H - 1)$ then $G' = K_H$ and G has $2H - 2$ vertices, again a contradiction. If $\mu(G') \geqslant 1/(H - 1)$ then Lemma 2.5 implies $\mu(G) \geqslant 1/(H - 1)$. On the other hand if $G(x) = K_{H-1}$ set $G' = G - N^*(x)$. If $\mu(G') < 1/(H - 1)$, $G' = K_H$ and G contains vertex disjoint copies of K_H and K_{H-1} contradicting Lemma 2.10. If $\mu(G') \geqslant 1/(H - 1)$, Lemma 2.4 forces $\mu(G) \geqslant 1/(H - 1)$.

THEOREM 4.2. *If G embeds on S_n $(n \geqslant 1)$ and G does not contain K_{H-1} then $\mu(G) > 1/(H - 1)$.*

PROOF. For $n = 1$ this is a corollary to Theorem 3.5. Suppose G is a graph with the fewest number of vertices with $\mu(G) \leqslant 1/(H - 1)$ and G not containing K_{H-1}. By Theorem 4.1 we know $\mu(G) = 1/(H - 1)$. Lemma 2.3 implies G contains a vertex x of degree less than $H - 1$. Set $G' = G - N^*(x)$. Since G' also does not contain K_{H-1} we know $\mu(G') > 1/(H - 1)$. Lemma 2.4 implies $\mu(G) > 1/(H - 1)$.

THEOREM 4.3. *If G embeds on S_n, $n \geqslant 5$, and $G \neq K_{H-1}$ and K_H, then $\mu(G) > 1/(H - 1)$.*

PROOF. Suppose G is a counterexample to the theorem with the fewest number of vertices; i.e. G is the smallest graph, not K_H or K_{H-1}, that has $\mu(G) < 1/(H-1)$. By Theorem 4.1 we may assume $\mu(G) = 1/(H-1)$. Also the theorem is obvious if $V \leq 2H-3$. If $V \geq 2H-2$ and $n \geq 5$ the average degree of G, $6 + 12(n-1)/V$, is less than $H-2$. Suppose x is a vertex of G of degree less than $H-2$. Set $G' = G - N^*(x)$. If $\mu(G') > 1/(H-1)$ we know $\mu(G) > 1/(H-1)$ by Lemma 2.4. If $\mu(G') \leq 1/(H-1)$, then $G' = K_{H-1}$ or K_H. If $G' = K_{H-1}$ then $V \leq 2H-3$. If $G' = K_H$, then $V = 2H-2$ and we know that $G(x)$ cannot form a K_{H-2} by Lemma 2.10. Thus the neighbors of x contain two vertices, say a and b, that are not adjacent. Consider the embedding of G'. There must be some vertex, say c, in G' with c not on the boundary of the region containing $G(x)$. If not, K_{H+1} would embed on S_n. The vertices a, b, and c form an independent set and $\mu(G) \geq 3/(2H-2) > 1/(H-1)$.

REMARK. Theorem 4.3 is valid for $n = 4$ though one must separately check graphs with $V = 2H - 2 = 18$ vertices. Since the theorem is also true for $n = 1$ one might suspect that it is true for all n. This is not the case as $K_7 + K_7$ embeds on S_2.

It would be nice to know more in general about the structure of graphs with small independence ratios. The difficulties involved are highlighted by J. The following gives some information.

THEOREM 4.4. *Given an integer m, let $N(m)$ be the smallest integer such that $N(m) \geq (2m + 11/2)^2$ and $H(N(m)) \geq \max\{m + 7, 2m\}$. If G embeds on S_n, $n \geq N(m)$, and $\mu(G) < 1/(H-m)$, then G contains K_{H-m+1}.*

PROOF. Suppose G is the smallest graph with $\mu(G) < 1/(H-m)$ and G does not contain K_{H-m+1}. We may assume $V > 2H-2m$. The average degree of G satisfies $6 + 12(n-1)/V < H - m$ when

$$12(n-1) < (H-m-6)V.$$

But

$$\begin{aligned} V(H-m-6) &\geq (2H-2m+1)(H-m-6) \\ &\geq H(2H-4m-11) \geq 3\sqrt{n}\cdot 4\sqrt{n} > 12(n-1). \end{aligned}$$

Thus the average degree is less than $H-m$. Let x be a vertex of degree less than $H-m$. Set $G' = G - N^*(x)$. If $\mu(G') \geq 1/(H-m)$ then Lemma 2.4 implies $\mu(G) \geq 1/(H-m)$. Thus $\mu(G') < 1/(H-m)$. Since G was the smallest graph with $\mu(G) < 1/(H-m)$ and not containing K_{H-m+1}, G' must contain K_{H-m+1} as does G, a contradiction.

V. $L(n)$; $n \geq 2$. We now change our point of view to show that almost all graphs embedded on a given surface have relatively large independence ratio. We begin by showing that in an important class of triangulations embedded on a given surface each graph contains a 6×6 configuration.

LEMMA 5.1. *Suppose G is embedded on S_n ($n \geqslant 2$) and G is a triangulation with no vertex of degree less than six. If $V > 21(n-1)$ then G contains a 6×6 configuration.*

PROOF. Assume that G contains no 6×6 configuration. Let V_i denote the number of vertices of G of degree i. Euler's Formula implies

$$2E = 6V + 12(n-1) = \sum_{i \geqslant 6} i\, V_i. \tag{1}$$

Thus

$$12(n-1) = \sum_{i \geqslant 7} (i-6)V_i \geqslant \sum_{i \geqslant 7} V_i = V - V_6.$$

Hence

$$V_6 \geqslant V - 12(n-1). \tag{2}$$

We now use a counting technique developed in [2]. Replace each edge of G by two directed edges, one in each direction. We count the number of edges in each of the following three categories:

N_1 = the number of edges entering a vertex of degree 6.

N_2 = the number of edges directed from a vertex of degree 6 to a vertex of degree greater than 6.

N_3 = the number of edges directed from a vertex of degree m to a vertex of degree m' where $m, m' > 6$.

Note that we have counted each directed edge at most once and thus each edge of G at most twice. We now obtain expressions for each of the N's. $N_1 = 6V_6$. If G contains no 6×6 configuration, then each vertex of degree 6 must be adjacent to at least four vertices of degree greater than 6. Thus $N_2 \geqslant 4V_6$. Similarly a vertex of degree m ($m \geqslant 7$) must be adjacent to at least four vertices of degree m' ($m' \geqslant 7$). Thus $N_3 \geqslant 4(V - V_6)$. Adding N_1, N_2, and N_3 we get

$$2E \geqslant 6V_6 + 4V_6 + 4(V - V_6) = 4V + 6V_6. \tag{3}$$

Combining (1), (3), and (2) yields

$$6V + 12(n-1) \geqslant 4V + 6V_6 \geqslant 4V + 6(V - 12(n-1)). \tag{4}$$

Simplifying gives $21(n-1) \geqslant V$. Thus if $V > 21(n-1)$, G must contain a 6×6 configuration.

THEOREM 5.2. *If G embeds on S_n, $n \geqslant 2$, then $\alpha(G) \geqslant (V - 21(n-1))/5$.*

PROOF. If the theorem is false, let n be the smallest genus of a surface for which it fails, and let G be the smallest graph which embeds on S_n and for which $\alpha(G) < (V - 21(n-1))/5$.

Suppose $hg(G) = 3$ and let C be the corresponding 3-cycle. Remove the vertices of C and all incident edges to obtain G', a graph which embeds on

S_{n-1}. If $n = 2$ Theorem 3.5 implies $\alpha(G') \geqslant (V-3)/5$ unless $G' = K_7, K_6, K_7 + K_4$, or J. If G' is one of these four, $\alpha(G) \geqslant (V-21)/5$ since V is less than 21. If $\alpha(G') \geqslant (V-3)/5$, $\alpha(G) \geqslant \alpha(G') \geqslant (V-3)/5 \geqslant (V-21)/5$. If $n > 2$ the minimality of n implies

$$\alpha(G) \geqslant \alpha(G') \geqslant (V - 3 - 21(n-2))/5 \geqslant (V - 21(n-1))/5.$$

Thus we may assume that $hg(G) > 3$. Suppose x is a vertex of G of degree less than five.

Set $G' = G - N^*(x)$. We know that $\alpha(G') \geqslant (V - 5 - 21(n-1))/5$. Thus $\mu(G') \geqslant 1/5 - 21(n-1)/(5(V-5))$. Applying Lemma 2.4 with $r = 5$, $s = 1/5 - 21(n-1)/(5(V-5))$ and $q = 1/5 - 21(n-1)/5V$ yields $\mu(G) \geqslant 1/5 - 21(n-1)/5V$ since $V = (1 - rs)/(q - s)$. Thus

$$\alpha(G) \geqslant (V - 21(n-1))/5.$$

Suppose G has a vertex x of degree 5. If $G(x) = K_6$ then $hg(G) = 3$, a case already considered. If $G(x) \neq K_6$ then x has two neighbors, a and b, that are not adjacent. Set $G' = G(x, a, b)$. Exactly as above Lemma 2.5 implies $\alpha(G) \geqslant (V - 21(n-1))/5$.

Thus we may assume that every vertex of G has degree at least six. As we are trying to show that G has suitably high independence ratio we may assume that G is a triangulation. By Lemma 5.1 G contains a 6×6 configuration. Since $hg(G) > 3$ and because of Lemma 2.9, we can apply Lemma 2.6 with $r \leqslant 10$, $s = 1/5 - 21(n-1)/(5(V-10))$ and

$$q = 1/5 - 21(n-1)/5V$$

to obtain $\mu(G) \geqslant 1/5 - 21(n-1)/5V$ since $V = (2 - rs)/(q - s)$.

Thus $\alpha(G) \geqslant (V - 21(n-1))/5$.

COROLLARY 5.3. *Suppose G embeds on S_n. Given $\varepsilon > 0$, if*

$$V \geqslant 21(n-1)/5\varepsilon,$$

then $\mu(G) \geqslant 1/5 - \varepsilon$.

COROLLARY 5.4. $L(n) \subset [\frac{1}{5}, 1]$, *for all* $n > 0$.

POSTSCRIPT. The validity of the conjecture mentioned in the introduction has now been established [3]. Also in [3] extensions of the results of this paper are reported; in particular the results of Theorems 4.2 and 4.3 have been extended in [8].

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